Vibrational modes and spectrum of oscillators on a scale-free network

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We study vibrational modes and spectrum of a model system of atoms and springs on a scale-free network in order to understand the nature of excitations with many degrees of freedom on the scale-free network. We assume that the atoms and springs are distributed as nodes and links of a scale-free network, assigning the mass M_i and the specific oscillation frequency ω_i of the i-th atom and the spring constant K_{ij} between the i-th and j-th atoms.

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I. INTRODUCTION

Recently there has been a notable progress in the study of the so-called scale-free network (SFN)[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In the point of view of network theory, the random network theory (RN) was first invented by Erdös and Rényi[11] and has been applied to many areas of sciences from physics such as Anderson localization[12], percolation[13], and energy landscape[14, 15] to biology such as the Kauffman's NK model[16]. Recently it was generalized to the small world network (SWN) models [17, 18, 19, 20, 21, 22, 23, 24, 25] and has been applied to physical systems such as percolation[17, 18, 19, 20, 21, 22, 23, 24, 25], arrays of coupled oscillators with synchronization known as the Kuramoto model[26], arrays of coupled lasers[27], arrays of the Josephson junctions[28], electric circuit[29], traffic transportation[30, 31], and protein folding[32]. And very recently, the SFN was discovered by studying the network geometry of the internet[1, 2, 3, 33, 34, 35, 36]. Albert, Jeong and Barabási[2, 3, 33, 34, 35, 36] opened up an area for studying very complex and growing network systems such as internet[17, 33, 34, 35, 36, 37], biological evolution[38], metabolic reaction[39], epidemic disease[40], human sexual relationship[41], and economy[42]. These are nicely summarized in the reviews by Barabási[2].

The nature of these SFNs is characterized by the power-law behavior of the distribution function for the number of nodes with k links such as $P(k) \propto k^{-\gamma}$ where $\gamma \approx 1-4$. In order to show the power-law distribution of the SFN, Albert and Barabási first proposed a very simple model called the Albert-Barabási (AB)'s SFN model[2, 3, 33, 34, 35, 36]. This system is constructed by the following process: Initially we put m_0 nodes as seeds for the system. Every time when a new node is added, m new links are distributed from the node to the existed nodes in the system with a preferential attachment probability $\Pi_i(k_i) = k_i/\sum_{i=1}^{N-1} k_i$, where k_i is the number of links at the i-th node and we have assumed $m \le m_0$. The development of this model is described by a continuum model $\frac{dk_i}{d\tau} = m\Pi_i(k_i) = \frac{mk_i}{2\tau}$. Then at time τ the system consists of $N(\tau)$ nodes and the $L(\tau)$ links with $L(\tau) = \frac{1}{2}\sum_{i=1}^{N(\tau)} k_i$. This model exhibits $\gamma = 3$ for the power-law. Thus, it has been concluded that the essential points of why a network grows to a SFN are attributed to the growth of the system and the preferential attachment of new nodes to old nodes existed already in the network.

However, although time evolution of the SFN has been intensively studied regarding nodes and links as metaphysical objects such as agents and relationships in an area of science, it seems that very few physical models putting real meaning on nodes and links in the SFNs have been studied in order to investigate excitations such as vibrations, phonons, and electrons, except diffusion[43] and spins[44] on the SFN and excitations in the RN[11, 12, 13, 14, 15, 16] and SWN[17, 18, 19, 20, 21, 22, 23, 24, 25]. So, we explore to study, as a prototype model, vibrational modes and spectrum of a system of atoms coupled by springs where the atoms and springs are located regarding as nodes and links of a SFN.

II. VIBRATIONAL MODEL

Let us introduce our vibrational model. We first adopt the AB's SFN model for the construction of a SFN and we regard nodes and links in the SFN as atoms and springs in our physical model. Assuming that q_i and ω_i are the displacement and the specific frequency of the i-th atom of mass M_i, respectively, we can define the hamiltonian of the system:

$$H = \sum_{i=1}^{N(\tau)} \left(\frac{M_i}{2}\dot{q}_i^2 + \frac{M_i\omega_i^2}{2} q_i^2 \right) + \sum_{i,j=1(i\neq j)}^{N(\tau)} \frac{K_{ij}}{2}(q_i - q_j)^2, \tag{1}$$

where $\dot{q}_i = \frac{dq_i}{dt}$ the velocity of the i-th atom and K_{ij} is the spring constant between the i-th and j-th atoms with $K_{ij} = K_{ji}$. Although we will concentrate to study only this atom-spring model in this paper, the generalizations of this model and the applications to other systems are straightforward. We expect that the physical nature of this model shares with those of such models and systems.

We now assume that the time interval $\Delta\tau$ for the development of link addition process is much larger than

that Δt of the physical model such that $|\Delta\tau| \gg |\Delta t|$. This guarantees that although the network grows in the course of its development, as long as the network consists of $N(\tau)$ nodes and $L(\tau)$ links, the vibrational model can be simultaneously solved. This means that *time evolution the network is adiabatic to the time motion of the atoms and springs*. By using the Euler-Lagrange equation, $\frac{d}{dt}(\frac{\partial H}{\partial \dot{q}_i}) = \frac{\partial H}{\partial q_i}$, we obtain

$$M_i(\ddot{q}_i + \omega_i^2 q_i) = \sum_{j=1}^{N(\tau)} K_{ji}(q_j - q_i), \tag{2}$$

for $i = 1, \ldots, N(\tau)$. Assuming $q_i(t) = q_i(\omega)e^{-i\omega t}$, Eq.(2) becomes

$$M_i(\omega_i^2 - \omega^2)q_i = \sum_{j=1}^{N(\tau)} K_{ji}(q_j - q_i), \tag{3}$$

for $i = 1, \ldots, N(\tau)$. This is the eigenequation for our system.

Let us assume that all springs are identical for the sake of simplicity such that $K_{ij} = K_0 A_{ij}$, where K_0 is the spring constant and A_{ij} is the ij-th component of the adjacency matrix $\hat{\mathbf{A}}$ for the network geometry. The components of the adjacency matrix are non-negative such that $A_{ij} = 0$ or 1 according to whether or not a link between the i-th and j-th nodes exist in the network. The link number $k_i(\tau)$ at the i-th atom (i.e., the order of the i-th node) is given by $k_i(\tau) = \sum_{j=1}^{N(\tau)} A_{ji}$. From this, the last term in Eq.(3) becomes $\sum_{j=1}^{N(\tau)} K_{ji}q_i = K_0 k_i(\tau)q_i$. Hence, in this setting, we obtain

$$\Omega_i q_i = K_0 \sum_{j=1}^{N(\tau)} A_{ji} q_j, \tag{4}$$

for $i = 1, \ldots, N(\tau)$, where

$$\Omega_i \equiv M_i(\omega_i^2 - \omega^2) + K_0 k_i(\tau). \tag{5}$$

III. GREEN'S FUNCTION FORMALISM

Let us now define the Green's function by

$$\sum_{j=1}^{N(\tau)} \left[\Omega_i \delta_{ij} - K_0 A_{ij}\right] G_{jk} = \delta_{ik}, \tag{6}$$

for i, $k = 1, \ldots, N(\tau)$, which is represented by $[\hat{\mathbf{G}}_0^{-1} - K_0\hat{\mathbf{A}}]\hat{\mathbf{G}} = \hat{\mathbf{1}}$ in the matrix representation where $\hat{\mathbf{1}}$ is the $N(\tau)\times N(\tau)$ unit matrix and $\hat{\mathbf{G}}_0$ is the $N(\tau)\times N(\tau)$ diagonal matrix defined by $\hat{\mathbf{G}}_0 = \Omega_i^{-1}\delta_{ij}$. Thus, the Green's function is formally obtained as $\hat{\mathbf{G}}^{-1} = \hat{\mathbf{G}}_0^{-1} - K_0\hat{\mathbf{A}}$. Furthermore, we can derive a series expansion of $\hat{\mathbf{G}}$ in terms of $\hat{\mathbf{G}}_0$ and $\hat{\mathbf{A}}$ as

$$\hat{\mathbf{G}} = \hat{\mathbf{G}}_0 + K_0\hat{\mathbf{G}}_0\hat{\mathbf{A}}\hat{\mathbf{G}}_0 + K_0^2\hat{\mathbf{G}}_0\hat{\mathbf{A}}\hat{\mathbf{G}}_0\hat{\mathbf{A}}\hat{\mathbf{G}}_0 + \cdots$$

$$= \hat{\mathbf{G}}_0 + K_0\hat{\mathbf{G}}_0\hat{\mathbf{A}}\hat{\mathbf{G}} = \hat{\mathbf{G}}_0 + \hat{\mathbf{G}}_0\hat{\mathbf{T}}\hat{\mathbf{G}}_0, \tag{7}$$

where $\hat{\mathbf{T}}$ is called the T-matrix defined as

$$\hat{\mathbf{T}} = K_0\hat{\mathbf{A}} + K_0^2\hat{\mathbf{A}}\hat{\mathbf{G}}_0\hat{\mathbf{A}} + \cdots = \hat{\mathbf{T}}_1 + \hat{\mathbf{T}}_2 + \cdots. \tag{8}$$

We can now derive the following:

$$(\hat{\mathbf{T}}_n)_{ij} = K_0^n(\hat{\mathbf{A}}\hat{\mathbf{G}}_0\hat{\mathbf{A}}\cdots\hat{\mathbf{G}}_0\hat{\mathbf{A}})_{ij}$$

$$= K_0^n \sum_{j_1, j_2, \cdots, j_{n-1}} \frac{A_{ij_1}A_{j_1 j_2}\cdots A_{j_{n-1}j}}{\Omega_{j_1}\Omega_{j_2}\cdots\Omega_{j_{n-1}}} \equiv K_0^n \Gamma_{ij}^{(n)}, \tag{9}$$

and since $K_{ij} = K_{ji}$ (i.e., $A_{ij} = A_{ji}$), we find

$$(\hat{\mathbf{T}}_n)_{ij} = (\hat{\mathbf{T}}_n)_{ji}. \tag{10}$$

From Eqs.(8) and (9), we find $(\hat{\mathbf{T}})_{ij} = K_0\Gamma_{ij}^{(1)} + K_0^2\Gamma_{ij}^{(2)} + \cdots$.

Let us consider the trace of the Green's function. We now get

$$Tr\hat{\mathbf{G}} = Tr(\hat{\mathbf{G}}_0 + \hat{\mathbf{G}}_0\hat{\mathbf{T}}\hat{\mathbf{G}}_0) = \sum_i \left(\frac{1}{\Omega_i} + \frac{(\hat{\mathbf{T}})_{ii}}{\Omega_i^2}\right). \tag{11}$$

We note here that if $M_i(\omega_i^2 - \omega^2) = 0$ such that $\Omega_i = K_0 k_i(\tau)$, then $(\hat{\mathbf{T}}_n)_{ij} = K_0 k_i P_{ij}$ with $k_i P_{ij} = k_j P_{ji}$, where $P_{ij}(\tau)$ means the probability that the walker starts at node i at time $t = 0$ and found at node j at time $t = \tau$ in terms of the language of the diffusion theory of Noh and Rieger [see Eqs(2) and (3) in [43]]. As is well-known, the density of states $\rho(\omega)$ is given by

$$\rho(\omega) = -\frac{1}{\pi} Tr\hat{\mathbf{G}}(\omega + i\epsilon). \tag{12}$$

Thus, the poles of the Green's function produce the spectrum of the system.

IV. SPECIAL LIMITS

Before going to do the direct calculation for spectrum of the system, let us consider some limits. (i) *The independent atom limit*. First, in the case of no springs of $K_0 = 0$, since $Tr(\hat{\mathbf{G}}) = Tr(\hat{\mathbf{G}}_0)$, the poles of the Green's function are given by $\Omega_j \equiv M_j(\omega_j^2 - \omega^2) = 0$, which trivially provides the discrete spectrum $\omega = \omega_j$ for $j = 1, \ldots, N(\tau)$. This means that the atoms independently vibrate with specific frequencies ω_i.

(ii) *The AB limit*. Second, in the case of very weak spring constant such as $K_0 \ll 1$, the the poles of the Green's function are obtained as

$$\omega^2 = \omega_j^2 + \frac{K_0}{M_j} k_j(\tau). \tag{13}$$

This means that each atom vibrates with frequency related to the number of links of the atom. Since the distribution of the nodes with k links is given by $P(k) \propto k^{-\gamma}$

in the SFN[2, 3], the distribution of the spectrum is given as

$$P(\omega^2 - \omega_j^2) \propto (\omega^2 - \omega_j^2)^{-\gamma}. \quad (14)$$

Hence, this limit shares with the nature of the AB's SFN geometry. Therefore, we may call this limit the AB limit.

(iii) *The localized mode limit.* Third, let us consider the limit of very small mass ($M_i \ll 1$) or very strong spring ($K_0 \gg 1$). In this case, we can ignore the frequency dependence in the eigenequation of Eq.(4) such as $\Omega_i \approx K_0 k_i$, which then yields

$$k_i q_i = \sum_{j=1}^{N(\tau)} A_{ji} q_j, \quad (15)$$

for $i = 1, \ldots, N(\tau)$. Since we can rewrite the above equation as $\sum_{j=1}^{N(\tau)} [k_i \delta_{ij} - A_{ij}] q_j = 0$, non-trivial solutions may exist only when the determinant $\det[k_i \delta_{ij} - A_{ij}]$ vanishes. This is realized when $q_j = q_i$ where j runs the adjacent links around the i-th atom. In this sense the mode is localized within the adjacent atoms.

V. CALCULATIONS OF THE SPECTRUM

Let us now calculate the spectrum of the system of oscillators in the SFN. This is carried out by directly diagonalizing Eq.(4). For the sake of simplicity, we assume that $M_i = M_0 = \text{const.}$ and $\omega_i = \omega_0 = \text{const.}$ and we adopt the AB-model for generating the SFN geometry. We have performed the calculations for the systems up to $N = 10^4$.

Fig.1 (a) shows the density of states of the system, where we have calculated for the case of $m = 2$ and $N = 10^4$ (blue) and the case of $m = 4$ and $N = 10^4$ (red), respectively, where we have used $m_0 = 5$. To obtain the distributions, we have used twenty configurations with different random numbers. The vertical axis means $\rho(\omega)\sqrt{\langle k \rangle_2}$, while the horizontal axis means $\omega^2/\sqrt{\langle k \rangle_2}$, where $\langle k \rangle_2$ stands for the second order average degree of a node [See Eq.(18)]. The shape of the curve is unique such that there is a peak at $\omega = \omega_0$ and the spectral tail exists in the whole range of the spectrum. This tendency means that there is a scale-free nature in the spectrum of the vibrational modes in the system.

Fig.1 (b) shows the tail behavior of the density of states. The density of states is shown in a log-log plot for the cases of the AB-model with $m = 2$ and $N = 10^4$ (blue circles), with $m = 4$ and $N = 7 \times 10^3$ (red crosses), and with $m = 4$ and $N = 10^4$ (red squares), respectively. The red line is a guide for showing $(\omega^2)^{-3}$.

From this we find that the tail behavior of Eq.(14) holds valid for the general cases as well. Therefore, we can conclude that our vibrational model shares common nature with the AB-model of the SFN. This is contrary to the conclusion previously obtained from the calculations of the spectrum of the adjacency matrix $\hat{\mathbf{A}}$ of the

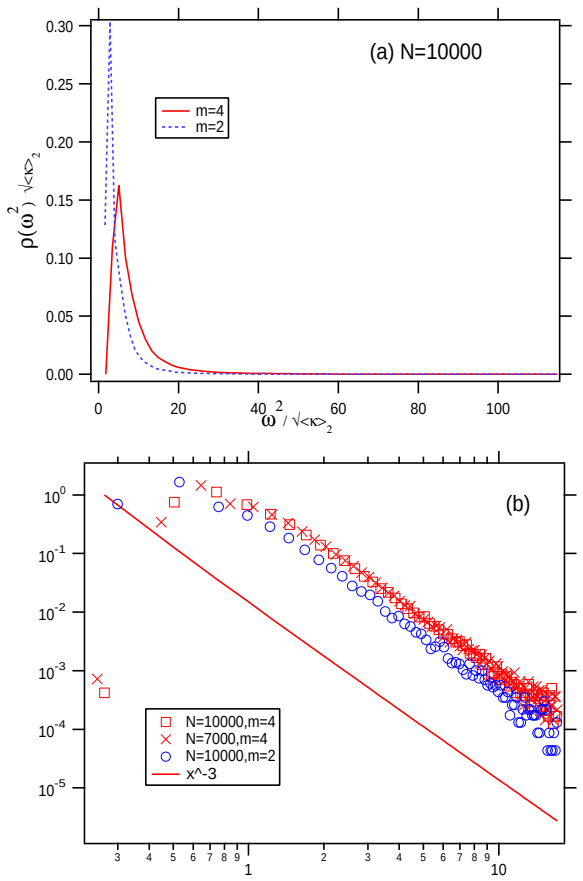

FIG. 1: (color online) The density of states and its tail behavior of the system of oscillators in the SFN. (a) The density of states is calculated for the cases of the AB-model with $m = 2$ (blue) and $m = 4$ (red) for $N = 10^4$, respectively. The distributions are obtained as an average over twenty configurations with different random numbers. The vertical axis means the density of states, $\rho(\omega^2)\sqrt{\langle k \rangle_2}$, while the horizontal axis means ω^2, where ω is the vibrational frequency of the oscillators and $\langle k \rangle_2$ the second order average degree of a node. (b) The tail behavior of the density of states is shown for the cases of the AB-model with $m = 2$ and $N = 10^4$ (blue circles), with $m = 4$ and $N = 7 \times 10^3$ (red crosses), and with $m = 4$ and $N = 10^4$ (red squares), respectively. The vertical axis means log-plot of the density of states while the horizontal axis means log-plot of ω^2, where ω is the vibrational frequency of the oscillators. The red line is a guide for showing $(\omega^2)^{-3}$. Here we have assumed that $\omega_0 = K_0 = M_0 = 1$ and $m_0 = 5$.

AB-model[45, 46, 47, 48]. There, when the network has the tail behavior of $P(k) \propto k^{-\gamma}$, the spectral tail for the eigenvalues λ of the adjacency matrix is given by $\rho(\lambda) \propto \lambda^{-\gamma'}$ where $\gamma' = 2\gamma - 1$. Therefore, since the AB-model has $\gamma = 3$, we conclude $\gamma' = 5$. This is different from our result of $\gamma' = 3$. The main reason for this phenomenon is explained as follows: In our vibrational model the Ω_i consists of the degree k_i of the node [see Eq.(4)]. Therefore, as the system grows, so does the magnitude of Ω_i. This can reduce the contributions of the adjacency matrix $\hat{\mathbf{A}}$ in the higher terms of the perturbation series

of Eq.(7). Hence, the spectral behavior is dominated by the pole of the unperturbed Green's function $\hat{\mathbf{G}}_0$. Thus, we are led to the same spectral behavior in the AB-limit.

The physical meaning of the above results can be understood as follows: The main peak in the density of states is attributed to vibrational modes with frequency ω_0. These modes are extremely localized within the least connected nodes in the SFN such that the total number of the localized modes provides the height of the peak. Since the number of modes is nothing but the number of degeneracy of the eigenequation, these localized modes are highly degenerate.

On the other hand, there is the power-law tail of $\rho(\omega^2) \propto (\omega^2)^{-3}$ as $\omega \to \infty$. This means that the larger the frequency of modes the fewer the number of modes. In other words, as the frequency is increasing, the number of modes is decreasing by the power-law. As the result, there appears only one mode with the maximum frequency (i.e., the maximum eigenvalue). The mode with the maximum frequency is extended over the entire system of the SFN. This situation means that in the SFN the lowest frequency modes can be very easily excited, but it is very hard to excite the maximum energy mode. Thus, the high frequency modes are very hard to exist in the system of oscillators coupled in the SFN. This nature is very different from that of the standard systems of networks such as RN[11] and lattices[13] that there are a small number of orders of nodes. This is the most prominent characteristic of our system.

VI. THE MAXIMUM EIGENVALUE

The behavior of the maximum eigenvalue λ_{max} of the adjacency matrix $\hat{\mathbf{A}}$ is very important in the network theory[1, 2, 45, 46, 47, 48]. In the standard networks such as the random networks[2, 11], the maximum eigenvalue λ_{max} cannot grow so fast as the network grows[1, 2, 45, 46, 47, 48]. And also, as in solid state physics, networks in most of physical systems provide the so-called energy band that is a spectrum with a finite region[12, 13]. This is due to the topology of the finite coordination number of atoms in the network of the lattice structure[12, 13]. So, in order to elucidate the difference between the SFNs and other networks the growth of the maximum eigenvalue is an important signature.

As was numerically studied by many authors[1, 2, 3, 45, 46, 47], the maximum eigenvalue λ_{max} of the adjacency matrix $\hat{\mathbf{A}}$ in the AB-model is proportional to $\sqrt{k_{max}}$ such that

$$\lambda_{max} \propto \sqrt{k_{max}}. \tag{16}$$

Here k_{max} means the maximum order of nodes in the network such that $k_{max} = \max_i\{k_i\}$ (We will use this notation for later purposes). And the numerical studies showed that $k_{max} \propto \sqrt{N}$. Therefore, we obtain

$$\lambda_{max} \propto N^{1/4}. \tag{17}$$

To see whether or not this is true in an arbitrary SFN and to know how general it is, very recently, Chung, Lu and Vu[48] have proved a very general theorem:

Theorem 1 *Suppose that the distribution of degrees of nodes in a SFN is represented by $P(k) \propto k^{-\gamma}$. Denote by $\langle k\rangle_2$ the second order average degree of a node. This is defined by*

$$\langle k\rangle_2 \equiv \frac{\sum_{i=1}^{N(\tau)} k_i^2}{\sum_{i=1}^{N(\tau)} k_i} = \frac{\langle k^2\rangle}{\langle k\rangle}, \tag{18}$$

where $\langle k^p\rangle = \frac{1}{N(\tau)}\sum_{i=1}^{N(\tau)} k_i^p$ with p integer. Then, (C1) if the exponent $\gamma > 2.5$, then

$$const.\langle k\rangle_2 \le \lambda_{max} \le const.\sqrt{k_{max}}. \tag{19}$$

(C2) If the exponent $2 < \gamma < 2.5$, then

$$const.\sqrt{k_{max}} \le \lambda_{max} \le const.\langle k\rangle_2. \tag{20}$$

(C3) And if the exponent $\gamma = 2.5$, then a transition happens.

We note here that in the paper of Chung, Lu and Vu[48] they used the notation $\tilde{d}$ for the second order average degree, instead of our notation $\langle k\rangle_2$ for it. Applying the above theorem to the AB-model of $\gamma = 3$, we find that the AB-model belongs to the first category. Hence, the theorem explains the numerical results[2, 3, 45, 46, 47].

In spite of such efforts, whether or not the growth of the maximum eigenvalue of a physical model on the SFN is not so well-known. This is because the eigenvalues of the adjacency matrix is different from those of the eigenequation of a physical system. In this sense, the problem to investigate the growth behavior of the maximum eigenvalue of the eigenequation of a physical system is a nontrivial problem. So, in order to see this point, let us consider the maximum eigenvalue ω_{max} of our vibrational system of oscillators.

We have performed calculations of the maximum eigenvalue (i.e., vibrational mode) ω_{max} in our model of oscillators on the AB-SFN, where $m = 4$ and N is developed up to $N = 10^4$. This is shown in Fig.2. The maximum eigenvalue ω_{max}^2 (circles), the maximum degree of a node k_{max} (triangles), and the second order average degree $\langle k\rangle_2$ of a node (+) are shown, respectively. Here we have obtained the following relation:

$$\omega_0^2 + 2\langle k\rangle_2 \le \omega_{max}^2 \le \omega_0^2 + 2k_{max}. \tag{21}$$

This looks similar to the result of Eq.(19) such that

$$\omega_{max}^2 \propto \omega_0^2 + \sqrt{k_{max}}. \tag{22}$$

However, this is not supported by our numerical calculations. Therefore, as the spectral tail of our vibrational model is different from that of the AB-model as discussed in the previous section, so is the growth behavior of the maximum eigenvalue of our vibrational model. This is an important character of our physical model with the AB-SFN.

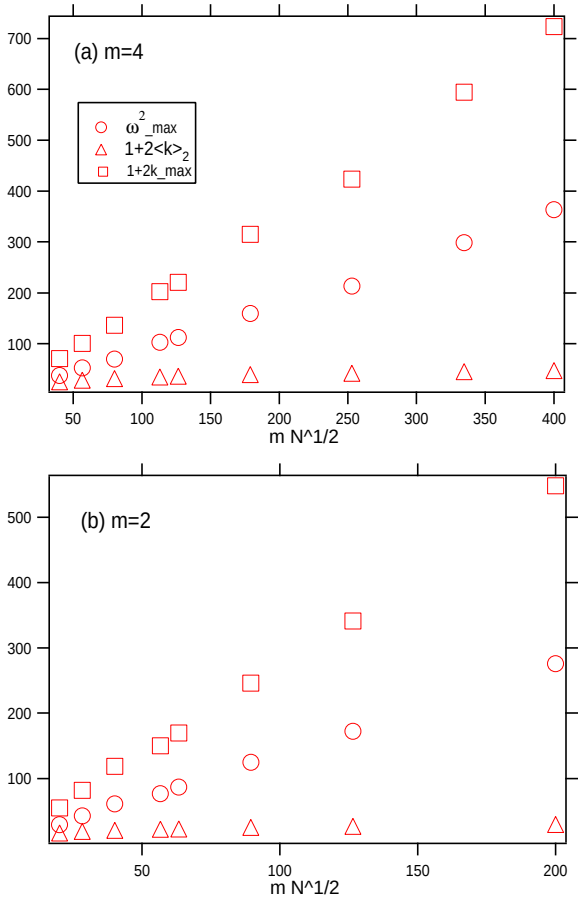

FIG. 2: (color online) The behavior of the maximum eigenvalue in the spectrum. The maximum eigenvalue ω^2_{max} (circles), the maximum degree of nodes $\omega_0^2 + 2\langle k\rangle_2$ (triangles), and the second order average degree $\omega_0^2 + 2k_{max}$ of nodes (squares) are shown in the vertical axis, respectively. The horizontal axis is scaled as $m\sqrt{N(\tau)}$. The calculations have been carried out for (a) the AB-model with $m = 4$ and N is up to $N = 10^4$, and for (b) the AB-model with $m = 2$ and N is up to $N = 10^4$. Here we have taken the values of $K_0 = M_0 = \omega_0 = 1$ and $\langle k\rangle_2$ stands for the second order average degree of a node.

VII. SOME THEOREMS ON THE MAXIMUM EIGENVALUE

Let us consider the origin of the inequality, Eq.(21). To see this, let us go back to Eq.(4). From the Hadamard-Gerschgorin's theorem[13] (see also Appendix A) we can derive an inequality

$$|\Omega_i| \leq K_0 \sum_{j=1}^{N(\tau)} |A_{ji}| \frac{|q_j|}{|q_i|}. \tag{23}$$

Since $|q_j|/|q_i| \leq 1$ and $|A_{ji}| = A_{ji}$, we can derive $|\Omega_i| \leq K_0 \sum_j |A_{ji}| = K_0 k_i(\tau)$, which then yields a theorem:

Theorem 2

$$\left|\omega^2 - \omega_i^2 - \frac{K_0}{M_i} k_i(\tau)\right| \leq \frac{K_0}{M_i} \sum_j |A_{ji}| = \frac{K_0}{M_i} k_i(\tau). \tag{24}$$

Thus, there exists at least one atomic site (i.e., node) that satisfies Eq.(24) for all eigenvalues ω. This implies that ω^2 is included within a disk of radius $\frac{K_0}{M_i}k_i(\tau)$ and its center $\omega_i^2 + \frac{K_0}{M_i}k_i(\tau)$.

Since in our model all K_{ij} (i.e., A_{ij}) are non-negative, by applying the Perron-Frobenius's theorem[13] (see also Appendix B) to Eq.(24) we can derive $|\omega^2 - \omega_i^2| \leq 2\frac{K_0}{M_0}k_i(\tau)$. Hence, the maximum frequency ω^2_{max} satisfies another theorem:

Theorem 3

$$\omega^2_{max} \leq \max_i \left[\omega_i^2 + 2\frac{K_0}{M_i} k_i(\tau)\right]. \tag{25}$$

And similarly we can obtain a more precise theorem:

Theorem 4

$$\min_i \left[\omega_i^2 + 2\frac{K_0}{M_i} k_i(\tau)\right] \leq \omega^2_{max} \leq \max_i \left[\omega_i^2 + 2\frac{K_0}{M_i} k_i(\tau)\right]. \tag{26}$$

From this, if we assume that $M_i \equiv M_0 = $ const. and $\omega_i = \omega_0 = $ const. and applying for the SFN, then Eq.(25) becomes

Theorem 5

$$2\frac{K_0}{M_0} \min_i\{k_i(\tau)\} \leq \omega^2_{max} - \omega_0^2 \leq 2\frac{K_0}{M_0} \max_i\{k_i(\tau)\}. \tag{27}$$

Hence, this theorem verifies our numerical results in the previous section. Therefore, the upper limit of the spectrum (i.e., spectral edge) grows as fast as the network grows. This is a remarkable fact for excitations in the SFN models and this nature is very different from that of Anderson localization where only mobility edge may appear in the spectrum and the band edge cannot grow as fast as the system size grows[12, 13].

The above Theorems 2-5 are good for the standard networks that the distribution of the orders of nodes is limited such as periodic lattice systems or the RN[13] or the SWM[17, 18, 19, 20, 21, 22, 23, 24, 25], since in these systems there exist finite lower and upper limits of the orders of nodes such that the error width is bounded as

$$\Delta(\omega^2_{max} - \omega_0^2) \geq \max_i[2\frac{K_0}{M_i}k_i(\tau)] - \min_i[2\frac{K_0}{M_i}k_i(\tau)]. \tag{28}$$

However, whether or not the above theorems can be accurate conditions for the SFN[2, 3] is not trivial, since in the SFNs there exist various orders of nodes without any bound but with the power-law distribution.

To study this point, we first observe that there is a particularly important nature of the adjacency matrix $\hat{\mathbf{A}}$

in the network theory. Denote by $\vec{k}_i = (A_{i1}, \ldots, A_{iN(\tau)})^t$ the i-th column vector of $\hat{\mathbf{A}}$. The vector represents the way of links between the i-th node and other linked nodes, such that it defines the order k_i of the i-th node such that

$$k_i \equiv \vec{k}_i^t \cdot \vec{k}_i = \sum_j A_{ij}. \tag{29}$$

Therefore, let us call $\vec{k}_i$ vectors the *link vectors*. Using this representation, we can rewrite the adjacency matrix as $\hat{\mathbf{A}} = (\vec{k}_1, \ldots, \vec{k}_{N(\tau)}) = (\vec{k}_1^t, \ldots, \vec{k}_{N(\tau)}^t)^t$, where t means the transpose. From this, we can derive that

$$\hat{\mathbf{A}}^2 = (\vec{k}_i^t \cdot \vec{k}_j), \tag{30}$$

which is a symmetric matrix and nothing but the Gramian matrix between the link vectors, $\vec{k}_i$, where

$$Tr(\hat{\mathbf{A}}^2) = \sum_{i=1}^{N(\tau)} \vec{k}_i^t \cdot \vec{k}_i = \sum_{i=1}^{N(\tau)} k_i = 2L(\tau). \tag{31}$$

Let us go back to Eq.(4). We now rewrite it as $\Omega_i \vec{q} = K_0 \hat{\mathbf{A}} \vec{q}$. Therefore, $\Omega_i^2 \vec{q} = K_0^2 \hat{\mathbf{A}}^2 \vec{q}$. Let us now use the Hadamard-Gerschgorin theorem[13] (Appendix A) or the Perron-Frobenius theorem[13] (Appendix B) for $K_0^2 \hat{\mathbf{A}}^2$, we can derive an inequality

$$|\Omega_i^2| \leq K_0^2 \sum_{j=1}^{N(\tau)} |(\hat{\mathbf{A}}^2)_{ji}| \frac{|q_j|}{|q_i|}. \tag{32}$$

Since $|q_j|/|q_i| \leq 1$ and $|(\hat{\mathbf{A}}^2)_{ji}| = (\hat{\mathbf{A}}^2)_{ji} = \vec{k}_j^t \cdot \vec{k}_i$, we can derive $|\Omega_i^2| \leq K_0^2 \sum_j |(\hat{\mathbf{A}}^2)_{ji}| = K_0^2 \sum_j \vec{k}_j^t \cdot \vec{k}_i = K_0^2 \vec{k}_{tot}^t \cdot \vec{k}_i$, where

$$\vec{k}_{tot}^t = \sum_{i=1}^{N(\tau)} \vec{k}_i^t = (k_1, k_2, \ldots, k_N). \tag{33}$$

Then we have

$$\left| \omega^2 - \omega_i^2 - \frac{K_0}{M_i} k_i(\tau) \right|^2 \leq \left(\frac{K_0}{M_i} \right)^2 \vec{k}_{tot}^t \cdot \vec{k}_i. \tag{34}$$

Therefore, it then yields a theorem:

Theorem 6

$$\left| \omega^2 - \omega_i^2 - \frac{K_0}{M_i} k_i(\tau) \right| \leq \frac{K_0}{M_i} \sqrt{\vec{k}_{tot}^t \cdot \vec{k}_i}. \tag{35}$$

Thus, there exists at least one atomic site (i.e., node) that satisfies Eq.(35) for all eigenvalues ω. This implies that ω^2 is included within a disk of radius $\frac{K_0}{M_i}\sqrt{\vec{k}_{tot}^t \cdot \vec{k}_i}$ and its center $\omega_i^2 + \frac{K_0}{M_i} k_i(\tau)$. Since $\left|\omega^2 - \omega_i^2\right| - \left|\frac{K_0}{M_i} k_i(\tau)\right| \leq \left|\omega^2 - \omega_i^2 - \frac{K_0}{M_i} k_i(\tau)\right|$, we obtain

$$\left|\omega^2 - \omega_i^2\right| \leq \frac{K_0}{M_i} \left(k_i(\tau) + \sqrt{\vec{k}_{tot}^t \cdot \vec{k}_i} \right). \tag{36}$$

Therefore, for the maximum frequency we obtain

$$\left|\omega_{max}^2 - \omega_i^2\right| \leq \max_i \left[\frac{K_0}{M_i} \left(k_i(\tau) + \sqrt{\vec{k}_{tot}^t \cdot \vec{k}_i} \right) \right]. \tag{37}$$

Hence, we obtain

$$\omega_{max}^2 \leq \omega_i^2 + \max_i \left[\frac{K_0}{M_i} \left(k_i(\tau) + \sqrt{\vec{k}_{tot}^t \cdot \vec{k}_i} \right) \right]. \tag{37}$$

Since $k_i \leq \vec{k}_{tot}^t \cdot \vec{k}_i$, we have $\max_i \left[\frac{K_0}{M_i} \left(k_i(\tau) + \sqrt{k_i(\tau)} \right) \right] \leq \max_i \left[\frac{K_0}{M_i} \left(k_i(\tau) + \sqrt{\vec{k}_{tot}^t \cdot \vec{k}_i} \right) \right]$. Therefore, the right hand of Eq.(37) is comparable with that of Eq.(26). In this way, Theorems 2-5 work for the SFN systems as well.

VIII. CONCLUSIONS

In conclusion, we have studied the system of oscillators connected by springs in the geometry of the AB SFN model. We first presented the Green function formalism for obtaining the spectrum of the vibrational modes of the system. In the case of very weak spring constant, using this formalism we find that the distribution of eigenmodes follows the same type of power-law distribution of degrees of a node in the AB model [see Eq.(13)] such that $P(\omega^2 - \omega_0^2) \propto (\omega^2 - \omega_0^2)^{-\gamma}$ with $\gamma = 3$. In the case of an arbitrary strength of spring constants, we have performed numerical calculations in order to obtain the spectrum of vibrational modes. We have found that even in this case, the distribution of eigenmodes obeys the same type of the power-law distribution of degrees of a node in the AB model as well [see Fig.1]. This is contrary to the distribution of eigenvalues of adjacency matrix in the AB model, where power-law distribution is given by $\rho(\lambda) \propto \lambda^{-\gamma'}$ with $\gamma' = 2\gamma - 1 = 5$.

This is a consequence of our model, where relative displacements between the individual oscillators are included in the Hamiltonian. This Hamiltonian provides the diagonal matrix elements in the eigenequation, which are proportional to the degrees of nodes [see Eq.(4)]. These diagonal elements can be regarded as on-site potentials in the problem. Since the degree of a node develops indefinitely, the on-site potential can be arbitrary large as the system is progressing. Therefore, the eigenvalues are strongly dominated by the magnitude of the diagonal elements of the eigenequation. Thus, the distribution of eigenmodes is affected by that of degrees of nodes such that the distribution of eigenmodes coincides with that of degrees of nodes in the network.

We finally have investigated the asymptotic behavior of the maximum eigenvalue ω_{max} of the system. We have found numerically that the maximum eigenvalue is bounded as in Eq.(21). From this, as the total number of nodes, N, is increasing, the maximum degree of

nodes becomes arbitrarily large. Therefore, the maximum eigenvalue can be arbitrarily large as $N \to \infty$. This coincides with the result of the maximum eigenvalue of adjacency matrix in the AB model. We have also proved the above numerical results by some mathematical theorems that are proved using the Hadamard-Gerschgorin theorem and the Perron-Frobenius theorem.

Thus we conclude that when we apply a certain physical model to the geometry of a SFN, the physical properties are strongly dominated by the nature of the SFN. In this sense, not only the network geometry of a SFN but also the property of physical models on a SFN are important in the study of the SFN. This direction will be very interesting for further researches.

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APPENDIX A: THE HADAMARD-GERSCHGORIN THEOREM

The following theorem is known as the Hadamard-Gerschgorin theorem in linear algebra[13]. Consider the following eigenequation:

$$(\lambda - h_i)q_i = \sum_{j=1(\neq i)}^{N} h_{ij}q_j, \qquad (A.1)$$

for $i = 1, \ldots, N$. Then, we find

$$|\lambda - h_i| \leq \sum_{j=1(\neq i)}^{N} |h_{ij}| \frac{|q_j|}{|q_i|}, \qquad (A.2)$$

for $i = 1, \ldots, N$. Since always $\frac{|q_j|}{|q_i|} \leq 1$, we obtain

$$|\lambda - h_i| \leq \sum_{j=1(\neq i)}^{N} |h_{ij}| \equiv B_i, \qquad (A.3)$$

for $i = 1, \ldots, N$. Now, we find a theorem that there exists at least one site such that the above equation Eq.(A.3) is valid for all λ. Eq.(A.3) means that λ is included within a disk of radius B_i with its center of h_i.

APPENDIX B: THE PERRON-FROBENIUS THEOREM

The following theorem is known as the Perron-Frobenius theorem in linear algebra[13]. Suppose that an $n \times n$ symmetric matrix H has all non-negative entries $h_{ij} \geq 0$. Then this satisfies an eigenequation $H|\psi_i\rangle = \lambda_i|\psi_i\rangle$. For any positive constants $c_1, c_2, \ldots, c_n$, the maximum eigenvalue $\lambda_{max}(H)$ satisfies

$$\lambda_{max}(H) \leq \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} \frac{c_j h_{ij}}{c_i} \right\}.$$

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[1] M. Faloutsos, P. Faloutsos, and C. Faloutsos, Compt. Commun. Rev. **29**, 251 (1999).
[2] Albert-László Barabási, *Linked*, (Penguin books, London, 2002) R. Albert and A.-L. Barabási, Statistical mechanics of complex networks, Rev. Mod. Phys. **74**, 47-97 (2002). A.-L. Barabási and E. Bonabeau, Scale-Free Networks, Scientific American (May), 60-69 (2003). References therein.
[3] R. Albert and A.-L. Barabási, Science **286**, 509 (1999). R. Albert and A.-L. Barabási, Phys. Rev. Lett. **85**, 5234 (2000). G. Bianconi and A.-L. Barabási, Phys. Rev. Lett. **86**, 5632 (2001).
[4] P. L. Krapivsky, S. Redner, and F. Leyvraz, Phys. Rev. Lett. **85**, 4629 (2000).
[5] P. L. Krapivsky, G. J. Rodgers, and S. Redner, Phys. Rev. Lett. **86**, 5401 (2001).
[6] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, Phys. Rev. Lett. **85**, 4633 (2000).
[7] S. N. Dorogovtsev and J. F. F. Mendes, Europhysics Lett. **52**, 33 (2000).
[8] L. Kullmann and J. Kertész, Phys. Rev. **E 63**, 051112 (2001).
[9] S. Bornholdt and H. Ebel, Phys. Rev. **E 64**, 035104 (2001).
[10] Z. Burda, J. D. Correia, and A. Krzywicki, Phys. Rev. **E 64**, 046118 (2001).
[11] P. Erdös and A. Rényi, Publ. Math. **6**, 290 (1959); Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960); Acta MAth. ACad. Sci. Hung. **12** 261 (1961). Erdös and A. Rényi theory; cf. B. Bollobás, *Random Graphs* (Academic Press, London, 1985).
[12] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[13] J. M. Ziman, *Models of Disorder*, (The Syndics of Cambridge University Press, Cambridge, 1982).
[14] D. J. Wales, M. A. Miller and T. R. Walsh, Nature **394**, 758 (1998).
[15] S. Sastry, P. G. Debendetti, and F. H. Stillinger, Nature **393**, 554 (1998).
[16] S. A. Kauffman, *The origins of Order*, (Oxford University, Oxford, 1993); *Investigations*, (Oxford University, Oxford, 2000).
[17] J. Kleinberg, Nature **406**, 845 (2000).
[18] D. J. Watts and S. H. Strogatz, Nature **393**, 440 (1998).
[19] M. E. J. Newman, C. Moore, and D. J. Watts, Phys. Rev.

Lett. **84**, 3201 (2000).
[20] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. **85**, 5468 (2000).
[21] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. **E 64**, 026118 (2001).
[22] M. Barthélémy and L. A. N. Amaral, Phys. Rev. Lett. **82**, 3180 (1999); (E) ibid. **82**, 5180 (1999).
[23] L. A. N. Amaral, A. Scala, M. Barthélémy, and H. E. Stanley, PNAS **97**, 11149 (2000).
[24] L. F. Lago-Fernández, R. Huerta, F. Corbacho, and J. A. Sigüenza, Phys. Rev. Lett. **84**, 2758 (2000).
[25] S. H. Strogatz, Nature **410**, 268, 276 (2001).
[26] H. Sakaguchi and Y. Kuramoto, Prog. Theor. Phys, **76**, 576 (1986).
[27] R.-D. Li and T. Erneux, Phys. Rev. **A 46**, 4252 (1992).
[28] K. Wiesenfeld, P. Colet, and S. H. Strogatz, Phys. Rev. Lett. **76**, 404 (1996).
[29] R. F. i Cancho, C. Janssen, and R. V. Solé, Phys. Rev. **E 64**, 046119 (2001).
[30] J. R. Banavar, A. Maritan and A. Rinaldo, Nature **399**, 130 (1999).
[31] J. R. Banavar, F. Colaiori, A. Flammini, A. Maritan and A. Rinaldo, Phys. Rev. Lett. **84**, 4745 (2000).
[32] A. Scala, L. A. N. Amaral, and M. Barthélémy, Europhys. Letts. **55**, 594 (2001).
[33] S. Lawrence and C. L. Giles, Science **280**, 98 (1998).
[34] B. A. Huberman, P. L. Pirolli, J. E. Pitkov and Rajan M. Lukose, Science **280**, 130 (1998).
[35] R. Albert, H. Jeong, and A.-L. Barabási, Nature **401**, 130 (1999).
[36] R. Albert, H. Jeong, and A.-L. Barabási, Nature **406**, 378 (2000).
[37] R. Cohen, K. Erez, D. ben-Abraham, and S. Halvlin, Phys. Rev. Lett. **85**, 4626 (2000). R. Cohen, K. Erez, D. ben-Abraham, and S. Halvlin, Phys. Rev. Lett. **86**, 3682 (2001).
[38] F. Slanina and M. Kotrla, Phys. Rev. Lett. **83**,5587 (1999). F. Slanina and M. Kotrla, Phys. Rev. **E 62**, 6170 (2000).
[39] H. Jeong, B. Tombor, R. Albert, Z. N. Oltval and A.-L. Barabási, Nature **407**, 651 (2000).
[40] R. Pastor-Santorras and A. Vespignani, Phys. Rev. Lett. **86**, 3200 (2001).
[41] F. Liljeros, C. R. Edling, L. A. N. Amaral, H. E. Stanley, and Y.Åberg, Nature **411**, 907 (2001).
[42] B. Arthur, Science **284**, 107 (1999).
[43] J. D. Noh and H. Rieger, Phys. Rev. Lett. **92**, 118701 (2004).
[44] A. Krawiecki, Int. J. Mod. Phys. **B 18**, 1759 (2004).
[45] I. J. Farkas, I. Derényi, A.-L. Barabási, and T. Vicsek, Phys. Rev. **E 64**, 026704 (2001).
[46] K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. **E 64**, 051903 (2001).
[47] S. N. Dorogovtsev, A. V. Goltsev, J. F. F. Mendes, and A. N. Samukhin, Phys. Rev. **E 68**, 026704 (2003).
[48] F. Chung, L. Lu, and V. Vu, Ann. Comb. **7**, 21 (2003); PNAS **100**, 6313 (2003).